Lecture 08: Graph Representation

Objective

- We shall develop a new graph representation to argue the security and correctness of cryptographic schemes
- As a representative application of this notation, we shall analyze private-key Encryption schemes using graphs

Assumption about Private-key Encryption Schemes

For simplicity of proof and clarity of intuition, we shall consider the class of all private-key encryption algorithms with the following restrictions

1. The key-generation algorithm Gen outputs a secret key sampled uniformly at random from the set $\mathcal{K}$
2. The encryption algorithm $\mathsf{Enc}_{\mathsf{sk}}(m)$ is deterministic

I want to emphasize that with a bit of effort, these *restrictions* can be removed

Graph of Private-key Encryption

Suppose $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is a private-key encryption scheme that satisfies the two restrictions we mentioned earlier. We construct the following bipartite graph

- The left partite set is the set of all message $\mathcal{M}$
- The right partite set is the set of all cipher-texts $\mathcal{C}$
- Given a message $m \in \mathcal{M}$ and a cipher-text $c \in \mathcal{C}$, we add an edge (m, c) labeled sk, if we have $c = \mathsf{Enc}_{\mathsf{sk}}(m)$

This is the graph corresponding to the encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$

Intuition. The edge labeled sk witnesses the fact that the message m is encrypted to the cipher-text c. Or, we write this as $m \xrightarrow{\mathsf{sk}} c$.
We emphasize that there might be more than one secret key that witnesses the fact that the message m is encrypted to the cipher-text c. Let $\mathsf{wt}(m, c)$ represent the number of secret keys sk such that sk witnesses the fact that c is an encryption of m

Describing Private-key Encryption Schemes

- Until now, we have represented private-key encryption scheme as a triplet of algorithms $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$
- Henceforth, we can equivalently express them as graphs

Property One: Characterization of Correctness

Theorem

A private-key encryption scheme (Gen, Enc, Dec) *is incorrect if and only if there are two distinct messages* $m, m' \in \mathcal{M}$*, a secret key* $\mathsf{sk} \in \mathcal{K}$*, and a cipher-text* $c \in \mathcal{C}$ *such that* $m \xrightarrow{\mathsf{sk}} c$ *and* $m' \xrightarrow{\mathsf{sk}} c$*.*

- Note that if there are two messages m, m' such that $m \xrightarrow{\mathsf{sk}} c$ and $m' \xrightarrow{\mathsf{sk}} c$, then Bob cannot distinguish whether Alice produced the cipher text c for the message m or m'. Hence, whatever decoding Bob performs, he is bound to be incorrect
- For the other direction, suppose Bob is unable to decode the (sk, c) correctly. If there is a unique $m \in \mathcal{M}$ such that $m \xrightarrow{\mathsf{sk}} c$, then Bob can obviously decode correctly. So, there must be two different messages $m, m' \in \mathcal{K}$ such that $m \xrightarrow{\mathsf{sk}} c$ and $m' \xrightarrow{\mathsf{sk}} c$

Property Two: Correct Schemes Cannot Compress I

Theorem

A correct private-key encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *has* $|\mathcal{C}| \geqslant |\mathcal{M}|$.

- Suppose not. That is, assume that we have a correct private-key encryption scheme with $|\mathcal{C}| < |\mathcal{M}|$.
- Fix any secret key $\mathsf{sk} \in \mathcal{K}$.
- Suppose $\mathcal{M} = \{m_1, m_2, \dots, m_\alpha\}$. Consider the following maps

$$m_1 \xrightarrow{\mathsf{sk}} c_1$$

$$m_2 \xrightarrow{\mathsf{sk}} c_2$$

$$\vdots$$

$$m_\alpha \xrightarrow{\mathsf{sk}} c_\alpha$$

Property Two: Correct Schemes Cannot Compress II

Note that these mappings exist because given any sk and m, the encryption algorithm maps to a unique ciphertext.

- Since $|\mathcal{C}| < |\mathcal{M}|$, by pigeon-hole principle there are two distinct messages $m, m' \in \mathcal{M}$ and a cipher text $c \in \mathcal{C}$ such that $m \xrightarrow{\mathsf{sk}} c$ and $m' \xrightarrow{\mathsf{sk}} c$
- So the scheme is incorrect. Hence contradiction.

Property Three: Characterization of Security I

Theorem

A private-key encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *is secure if and only if for any c and two distinct messages* $m, m' \in \mathcal{M}$ *we have* $\mathrm{wt}(m, c) = \mathrm{wt}(m', c)$.

- For any $m \in \mathcal{M}$ and $c \in \mathcal{C}$, note that we have $\mathbb{P}\left[\mathbb{C} = c | \mathbb{M} = m\right] = \mathrm{wt}(m, c)/|\mathcal{K}|$.
- Exercise: Prove that the security definition we have studied is equivalent to saying the following

  "For any two distinct messages $m, m' \in \mathcal{M}$ and a cipher-text $c \in \mathcal{C}$ we have: $\mathbb{P}\left[\mathbb{C} = c | \mathbb{M} = m\right] = \mathbb{P}\left[\mathbb{C} = c | \mathbb{M} = m'\right]$"
- Given this result, we can conclude that a scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is secure if and only if

  "For any two distinct messages $m, m' \in \mathcal{M}$ and a cipher-text $c \in \mathcal{C}$ we have: $\mathrm{wt}(m, c) = \mathrm{wt}(m', c)$"

Property Three: Characterization of Security II

- **Food for thought.** In a secure scheme, if there are $m \xrightarrow{\mathsf{sk}} c$, then for all $m' \in \mathcal{M}$ there exists some sk' such that $m' \xrightarrow{\mathsf{sk}'} c$
- **Food for thought.** The size of the set $\mathcal{K}$ need not be divisible by the size of the set $\mathcal{M}$. However, if there is a message m and a cipher-text c such that $\mathsf{wt}(m, c) = w$, then the number of secret keys $|\mathcal{K}| \geqslant w|\mathcal{M}|$. Why?

Property Four: Correct+Secure Schemes need Lots of Keys I

Theorem

A correct and secure private-key encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *has* $|\mathcal{K}| \geqslant |\mathcal{M}|$

- Suppose not. That is, there is a correct and secure scheme with $|\mathcal{K}| < |\mathcal{M}|$.
- Fix a cipher-text $c \in \mathcal{C}$ such that there exists $m \in \mathcal{M}$ and $\mathsf{sk} \in \mathcal{K}$ such that $m \xrightarrow{\mathsf{sk}} c$. Intuitively, we are picking a ciphertext that has a positive probability. For example, we are not picking a ciphertext that is never actually produced.
- Let the message space be $\mathcal{M} = \{m_1, m_2, \dots, m_\alpha\}$
- Note that, for any $m_i \in \mathcal{M}$ there exists some sk_i such that $m_i \xrightarrow{\mathsf{sk}_i} c$ (This is a property of secure private-key encryption schemes that was left as an exercise in the previous slide)

Property Four: Correct+Secure Schemes need Lots of Keys II

- Now, consider the mappings

$$
\begin{aligned}
m_1 &\xrightarrow{\mathsf{sk}_1} c \\
m_2 &\xrightarrow{\mathsf{sk}_2} c \\
&\vdots \\
m_\alpha &\xrightarrow{\mathsf{sk}_\alpha} c
\end{aligned}
$$

- Since $|\mathcal{K}| < |\mathcal{M}|$, by pigeon-hole principle, there exists two distinct messages m_i, m_j such that $\mathsf{sk}_i = \mathsf{sk}_j$ in the above mappings.
- This violates correctness. Hence contradiction.

Optimality of One-time Pad

- Note that any correct private-key encryption scheme must have $|\mathcal{C}| \geqslant |\mathcal{M}|$ (property two)
- Note that any correct and secure private-key encryption scheme must have $|\mathcal{K}| \geqslant |\mathcal{M}|$ (property four)
- One-time pad is a correct and secure scheme that achieves $|\mathcal{K}| = |\mathcal{C}| = |\mathcal{M}|$

Additional Food for Thought

- Recall that Property four states that the "correctness and security" of a private-key encryption scheme implies that the size of the set of keys is greater-than-or-equal to the size of the set of messages. For any $\mathcal{M}$, construct a correct but <u>insecure</u> private-key encryption scheme such that $|\mathcal{K}| = 1$! This result shall show the necessity of both correctness and security in that property.
- Another natural question is: Can we provide such guarantees for private-key encryption schemes that are secure <u>but incorrect</u>? The answer is NO. Think of a private-key encryption scheme that is secure (but incorrect) and works for any message set $\mathcal{M}$ and has $|\mathcal{K}| = |\mathcal{C}| = 1$!